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RESEARCH ARTICLE

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Fixed point theory and nonexpansive mappings

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Abstract Recall that a Banach space X has the weak fixed point property if for any nonempty weakly compact subset C of X and any nonexpansive mapping $T : C \to C$, T has at least one fixed point. In this article, we present three recent results using the ultraproduct technique. We also provide some open problems in this area.

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الملخص

استذكر أن أي فضاء باناخ X يمتلك خاصية النقطة الثابتة الضعيفة إذا كانت هناك نقطة ثابتة واحدة على الأقل لأي راسم غير تمددي $T: C \to C$، حيث C أية مجموعة جزئية غير خالية من X ومتراصة بشكل ضعيف. نقدم في هذا البحث ثلاث نتائج جديدة باستخدام تقنية فوق الجداء. نقدم أيضاً بعض المسائل المفتوحة في هذا المجال.

1 Introduction

For any closed bounded and convex subset C of a Banach space X, let r_C be the function from C to $\mathbb{R}$ defined by

$$r_C(x) = \sup\{\|y - x\| : y \in C\}.$$

Then r_C is a continuous convex function. The radius $r(C)$ and diameter $\mathrm{diam}(C)$ of C are defined by

$$r(C) = \inf\{r_C(x) : x \in C\} \quad \text{and} \quad \mathrm{diam}(C) = \sup\{r_C(x) : x \in C\}.$$

A map $T : C \to C$ is said to be *nonexpansive* if

$$\|Tx - Ty\| \leq \|x - y\|$$

for all $x, y \in C$. C is said to have the *fixed point property* (for nonexpansive maps) if for any nonexpansive map $T : C \to C$, there is $x \in C$ such that $Tx = x$. X is said to have the (*weak*) *fixed point property* if every (weakly compact) closed, bounded, convex subset C of X has the fixed point property. A nonempty, closed, convex, T-invariant subset K of C is said to be *minimal* if K_1 is a nonempty, closed, convex, T-invariant subset of K, then $K_1 = K$. By Zorn's lemma, for any weakly compact convex subset C and any nonexpansive map $T : C \to C$, C has a minimal (T-invariant) subset. Suppose that the closed convex hull, $\overline{\mathrm{co}}(T(C)) = C$. Then for any $x \in C$,

$$\begin{aligned} r_C(x) &= \sup\{\|x - y\| : y \in C\} \geq \sup\{\|T x - T y\| : y \in C\} \\ &= \sup\{\|T x - z\| : z \in \overline{\mathrm{co}}(C)\} \\ &= \sup\{\|T x - z\| : z \in C\} = r_C(T x). \end{aligned}$$

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Therefore, if $\beta > r(C)$, then the set $\{x \in C : r_C(x) \le \beta\}$ is a nonempty, closed, convex, T-invariant subset of C. If C is a minimal T-invariant set, then $\overline{\mathrm{co}}(T(C)) = C$. We have proved the following lemma.

Lemma 1.1 *Let C be a closed bounded and convex subset of a Banach space X and T a nonexpansive map from C into itself.*

(1) *Suppose that $\overline{co}(T(C)) = C$ and $\beta > r(C)$. Then the set*

$$\{x \in C : r_C(x) \le \beta\}$$

is a nonempty, closed, convex, T-invariant subset of C.

(2) *(Kirk) Suppose that C is minimal. Then r_C is a constant function and $r(C) = \mathrm{diam}(C)$.*

Let C be a bounded, closed, convex subset of a Banach space and $T : C \to C$ a nonexpansive mapping. A sequence (x_n) in C is said to be an *approximate fixed point sequence* for T if

$$\lim_{n\to\infty} \|T x_n - x_n\| = 0.$$

By Banach's contraction principle, if T is a nonexpansive map from a bounded closed and convex subset C of a Banach space X into itself, then T has an approximate fixed point sequence. The following lemma is due to Karlovitz [11] and Goebel [8].

Lemma 1.2 (*Karlovitz–Goebel Lemma*) *Let C be a nonempty, closed, bounded, and convex subset of a Banach space, and assume that $T : C \to C$ is nonexpansive. Let (x_n) be an approximate fixed point sequence for T and let $\psi : C \to \mathbb{R}^+$ be the function defined by $\psi(x) = \limsup_{n\to\infty} \|x - x_n\|$. Let $\alpha = \inf\{\psi(x) : c \in C\}$.*

(1) *Then for any $\beta > \alpha(\beta \ge \alpha$ if there is x such that $\psi(x) = \alpha)$, the set*

$$K = \{x : \psi(x) \le \beta\}$$

is a nonempty, closed, convex, T-invariant subset of C.

(2) *Suppose that C is minimal and weakly compact. Then $\psi(x) = diam(C)$ for all $x \in C$.*

Proof It is easy to see that ψ is a continuous convex function. If $\psi(x) \le \gamma$, then

$$\psi(T x) = \limsup_{n\to\infty} \|T x - x_n\| = \limsup_{n\to\infty} \|T x - T x_n\| \le \limsup_{n\to\infty} \|x - x_n\| = \psi(x).$$

So if $\beta > \alpha$, then the set $K = \{x : \psi(x) \le \beta\}$ is a nonempty, closed, convex, T-invariant subset of C. We have proved (1).

It is clear that for any $x \in C, \psi(x) \le \mathrm{diam}(C)$. Assume that C is minimal. By (1), ψ is constant. By passing to a subsequence of (x_n), we may assume that (x_n) converges to y weakly. By Lemma 1.1 (2), $r_C(y) = \mathrm{diam}(C)$. For any $\epsilon > 0$, there is $z \in C$ such that $\|z - y\| \ge \mathrm{diam}(C) - \epsilon$. Then for any $x \in C$,

$$\psi(x) = \psi(z) = \limsup_{n\to\infty} \|x_n - z\| \ge \|y - z\| \ge \mathrm{diam}(C) - \epsilon.$$

We have proved that $\psi(x) = \mathrm{diam}(C)$ for any $x \in C$. The proof is complete. □

Remark 1.3 Let C be a weakly compact, convex subset of a Banach space and $T : C \to C$ is nonexpansive. Suppose that C is minimal and $\mathrm{diam}(C) = 1$. Lemma 1.2 (2) is equivalent to the following statement:

For any $x \in C$ and $\epsilon > 0$, there is $\delta > 0$ such that if $y \in C$ and $\|y - T y\| < \delta$, then $\|x - y\| \ge 1 - \epsilon$.

It has been conjectured that every Banach space has the weak fixed point property. But Alspach [2] gave the following counterexample.

Example 1.4 Let C be the set

$$C = \left\{ f \in L_1[0, 1] : 0 \le f \le 1 \quad \text{and} \quad \int_0^1 f(t)\,\mathrm{d}t = \frac{1}{2} \right\}.$$

Let T be the map from C to C defined by

$$T\ f(t) = \begin{cases} 2f(2t) \wedge 1 & 0 \le t < \frac{1}{2}, \\ (2f(2t-1) \vee 1) - 1 & \frac{1}{2} \le x < 1. \end{cases}$$

It is easy see that T is an (into) isometry and T has no fixed point.

Maurey introduced the ultraproduct technique in the study of this problem and proved the following results [6,17]:

(1) c_0 has the weakly fixed point property.
(2) Every reflexive subspace of L_1 has the fixed point property.
(3) Every superreflexive Banach space with a 1-unconditional basis has the weak fixed point property.
(4) Let C be a nonempty, bounded, closed, convex subset of a superreflexive Banach space X and $T : C \to C$ an isometry (i.e., $\|x - y\| = \|Tx - Ty\|$ for all $x, y \in C$). Then T has a fixed point.

In this article, we consider the following three problems:

Problem 1 *Let (P) be a geometric property of Banach spaces. Does X have the weak fixed property if X has property (P)?*

Problem 2 *Let $(X, \|\cdot\|)$ be a Banach space. Is there an equivalent norm $|\cdot|$ of X such that $(X, |\cdot|)$ has the weak fixed point property?*

Problem 3 *Let $(X, \|\cdot\|)$ be a Banach space with the weak fixed point property. Find a constant $C > 1$ such that if Y is a Banach space with the Banach–Mazur distance $d(X, Y) < C$, then Y has the fixed point property.*

It is known that every uniformly convex Banach space (or Banach space with normal structure) has the weak fixed point property. On the other hand, we do not know whether every reflexive (or superreflexive) Banach space has the fixed point property or not. Let X be a Banach space. A subset A of X is *symmetrically ϵ-separated* if the distance between any two distinct points of $A \cup (-A)$ is at least ϵ. X is said to be *O-convex* if the unit ball of X contains no symmetrically $(2-\epsilon)$-separated subset of cardinality n for some $\epsilon > 0$ and some $n \in \mathbb{N}$. For any $\epsilon > 0$ a convex subset A of the unit ball of X is an *ϵ-flat* if for any $a \in A$, $\|a\| > 1 - \epsilon$. A collection $\mathcal{D}$ of the unit ball X is said to be *jointly complemented* if, for each distinct ϵ-flats A and B in $\mathcal{D}$, the set $A \cap (B \cup (-B))$ is empty. Let

$$E(n, X) = \inf\{\epsilon : \text{the unit ball of } X \text{ contains a jointly complemented collection of } \epsilon\text{-flats of cardinality } n\}.$$

X is said to be *E-convex* if $E(n, X) > 0$ for some $n \in \mathbb{N}$ [19]. It is known that every O-convex Banach space X is superreflexive, and a Banach space is O-convex if and only if X^* is E-convex. In Sect. 2, we introduce the ultraproduct (in the Sims' sense) [22] and show that if X is E-convex, then X has the weak fixed point property.

Partington [20,21] has proved that every renorming of $\ell_\infty(\Gamma)$ for Γ uncountable and any renorming of ℓ_∞/c_0 contain an isometric copy of ℓ_∞. Thus, if Y is a Banach space that is isomorphic to $\ell_\infty(\Gamma)$ for some uncountable set Γ or ℓ_∞/c_0, then Y does not have the weak fixed point property. On the other hand, it is known that every separable Banach space can be renormed to have normal structure. Therefore, every separable Banach space has an equivalent norm to satisfy the weak fixed point property. It would be interesting to identify some classes of Banach spaces which can be renormed to satisfy the weak fixed point property. Let X be a Banach space. Let $\mathcal{A}$ be the set of all weak null sequence (x_n) in the unit ball of X. García-Falset's coefficient $R(X)$ is defined by

$$R(X) = \sup\{\liminf_{n\to\infty} \|x_n + x\| : \|x\| = 1\}.$$

It is known that for any Banach space X, X has the weak fixed point property if $R(X) < 2$. Let X, Y be two Banach spaces such that $R(Y) < 2$. In Sect. 3, we show that if there is a one-to-one bounded linear map from X to Y, then X has an equivalent norm which satisfies the weak fixed point property [4]. It is known that $R\big(c_0(\Gamma)\big) = 1$, and for any reflexive Banach space X, there are Γ and a bounded one-to-one linear map from X to $c_0(\Gamma)$. Thus, every reflexive Banach space has an equivalent norm that satisfies the weak fixed point property.

Suppose that X has the Schur property. Then every weakly compact subset of X is compact. By the Schauder fixed point theory, every Banach space with the Schur property has the weak fixed point property. We do not know whether there is a Banach space X such that X does not have the Schur property and every equivalent norm on X satisfies the weak fixed point property. It is interesting to find a constant C such that for any Banach space X, if $d(\ell_2, X) < C$, then X has the fixed point property. In Sect. 4, we show that C can be chosen as $\frac{\sqrt{5+\sqrt{17}}}{2}$ [18].

In Sect. 5, we will give some remarks and open questions. For more details of the fixed point property, we refer to [1,12,22]

2 The space $\ell_\infty(X)/c_0(X)$

For any Banach space X and any free ultrafilter $\mathcal{U}$ of $\mathbb{N}$, let

$$\ell_\infty(X) = \left\{(y_n) : y_n \in X \text{ and } \|(y_n)\|_{\ell_\infty(X)} = \sup_n \|y_n\| < \infty\right\};$$
$$c_0(X) = \left\{[y_n] \in \ell_\infty(X) : \lim_{n\to\infty} \|y_n\| = 0\right\};$$
$$\mathcal{N} = \left\{[y_n] \in \ell_\infty(X) : \lim_{n\to\mathcal{U}} \|y_n\| = 0\right\}.$$

We shall denote the quotient space, $\ell_\infty(X)/c_0(X)$ (respectively, $\ell_\infty(X)/\mathcal{N}$), by $\widetilde{X}$ (respectively, $\widehat{X}$). ($\widehat{X}$ is called the ultrapower of X with respect to $\mathcal{U}$.) Let C be a weakly compact, convex subset of a Banach space X and let $T : C \to C$ be a nonexpansive map without any fixed point. Without loss of generality, we may assume that C is a minimal T-invariant subset such that $\text{diam}(C) = 1$. Let $\widetilde{C}$ (respectively, $\widehat{C}$) be the set

$$\widetilde{C} = \{[x_n] \in \widetilde{X} : x_n \in C\};$$
$$(\text{respectively, } \widehat{C} = \{[x_n] \in \widehat{X} : x_n \in C\}).$$

Let (y_n) and (z_n) be two sequences in C so that (y_n) and (z_n) belong to the same equivalence class. Then $\lim_{n\to\infty} \|y_n - z_n\| = 0$. But T is uniformly continuous.

$$\lim_{n\to\infty} \|T\,y_n - T\,z_n\| = 0.$$

Let $\widetilde{T}$ be the mapping from $\widetilde{C}$ to $\widetilde{C}$ defined by $\widetilde{T}[y_n] = [T\,y_n]$. We have shown that $\widetilde{T}$ is well-defined. It is easy to see that $\widetilde{T}$ is nonexpansive. Let I be the map from C to $\widetilde{C}$ defined by $I(x) = [x]$ (the constant sequence x). It is easy to see that I is an isometry and $I(T\,x) = [T\,x]$. We shall denote the constant sequence $[x](x \in C)$ by x. By the Karlovitz-Goebel Lemma, we have the following lemma:

Lemma 2.1 *Let C be a weakly compact convex subset of a Banach space and $T : C \to C$ a nonexpansive map. Assume that C is minimal and $\text{diam}(C) = 1$. Then*

(1) (*Maurey* [17]) *A point $[x_n]$ is a fixed point of $\widetilde{T}$ if and only if (x_n) is an approximate fixed point sequence for T (so $\widetilde{T}$ has at least one fixed point). If $[x_n]$ is a fixed point for $\widetilde{T}$, then $\|[x_n] - x\| = 1$ for all $x \in C$.*
(2) (*Lin* [14]) *If $\mathbf{w}_n$ is an approximate fixed point sequence for $\widetilde{T}$, then for any $x \in C$,*

$$\lim_{n\to\infty} \|\mathbf{w}_n - x\|_{\widetilde{X}} = 1.$$

Let (x_n) be a weakly convergent, approximate fixed point sequence for T. By translation, we may assume that (x_n) converges to 0 weakly (so $0 \in C$). Fix $0 < t < 1$ and let

$$W_1 = \{t[x_n]\};$$
$$W_k = \text{co}(W_{k-1} \cup \widetilde{T}(W_{k-1})) \quad \text{if } k \geq 2;$$
$$\widetilde{W_t} = \overline{\bigcup_{k=1}^{\infty}} W_k.$$

Then $\widetilde{W_t}$ is the smallest invariant closed convex subset of $\widetilde{C}$ of $\widetilde{T}$ which contains $t[x_n]$. We need the following lemma [15].

Lemma 2.2 *For any $0 < t < 1$, let $[w_n]$ be an element in $\widetilde{W_t}$.*

(1) $\lim_{n\to\infty} \|x_n - w_n\| = 1 - t$.
(2) *For any $x \in C$,* $\liminf_{n\to\infty} \|w_n - x\| \geq t$.
(3) *If w is a weak limit point of (w_n), then* $\lim_{n\to\infty} \|w_n - w\| = t$.
(4) *for any subsequence (w_{n_k}) of (w_n),*

$$\limsup_{k\to\infty} \limsup_{j\to\infty} \|w_{n_k} - w_{n_j}\| = t.$$

(5) *For any weak limit point w of (w_n),*

$$\|w\| \le 1 - t,$$
$$\liminf_{n\to\infty} \|x_n - w_n + w\| \ge 1 - t.$$

Proof By Lemmas 1.2 and 2.1, we have

$$\|[x_n] - t[x_n]\| = (1-t)\|[x_n]\| = 1 - t,$$
$$\lim_{n\to\infty}\lim_{m\to\infty} \|tx_n - tx_m\| = t \lim_{n\to\infty}\lim_{m\to\infty} \|x_n - x_m\| = t \lim_{n\to\infty} 1 = t,$$
$$\lim_{n\to\infty} \|tx_n - 0\| = \lim_{n\to\infty} t\|x_n - 0\| = t.$$

Let $[w_n]$ be an element in $\widetilde{W_t}$. We claim that

(a) $\limsup_{n\to\infty} \|x_n - w_n\| = \|[x_n] - [w_n]\| \le 1 - t$.
(b) $\limsup_{n\to\infty} \limsup_{m\to\infty} \|w_n - w_m\| \le t$.
(c) If w is a weak limit point of (w_n), then $\limsup_{n\to\infty} \|w_n - w\| \le t$.

(a) follows from the fact that the intersection of $\widetilde{C}$ and the closed ball in $\widetilde{X}$ centered at $[x_n]$ of radius $1-t$ is invariant under $\widetilde{T}$ and it contains $t[x_n]$.

Proof of (b). Note that T is nonexpansive. For any sequence (y_n) in C,

$$\limsup_{n\to\infty}\limsup_{m\to\infty} \|Ty_n - Ty_m\| \le \limsup_{n\to\infty}\limsup_{m\to\infty} \|y_n - y_m\|.$$

If (b) holds for a subset $\widetilde{D}$ of $\widetilde{C}$, then (b) holds for the convex hull $\mathrm{co}(\widetilde{D} \cup \widetilde{T}(\widetilde{D}))$. Note that (b) holds for $W_1 = \{t[x_n]\}$. By induction, (b) holds for all $W_{k+1} = \mathrm{co}(W_k \cup \widetilde{T}(W_k))$. Since $\widetilde{W_t} = \overline{\bigcup_{k=1}^{\infty} W_k}$, (b) holds for $\widetilde{W_t}$.

Proof of (c). Let $[w_n]$ be any element in $\widetilde{W_t}$ and let w be a weak limit point of (w_n). Since C is weakly compact, there is a subsequence (w_{n_k}) of (w_n) that converges to w weakly. Then

$$\begin{aligned}\limsup_{m\to\infty} \|w_m - w\| &\le \limsup_{m\to\infty}\limsup_{k\to\infty} \|w_m - w_{n_k}\| \\ &\le \limsup_{m\to\infty}\limsup_{n\to\infty} \|w_m - w_n\| \le t.\end{aligned}$$

We have proved (c).

Proof of (1). Suppose that (1) does not hold. By (a), there exist $[w_n] \in \widetilde{W_t}$ and a subsequence (w_{n_k}) of (w_n) such that

$$\lim_{k\to\infty} \|x_{n_k} - w_{n_k}\| \text{exists} \quad \text{and} \quad \lim_{k\to\infty} \|x_{n_k} - w_{n_k}\| < 1 - t.$$

By (c), for any weak limit point w of (w_n), $\limsup_{n\to\infty} \|w_n - w\| \le t$. Then

$$\limsup_{k\to\infty} \|x_{n_k} - w\| \le \lim_{k\to\infty} \|x_{n_k} - w_{n_k}\| + \limsup_{k\to\infty} \|w_{n_k} - w\| < 1.$$

This contradicts the Karlovitz-Goebel Lemma.

Proof of (2). Suppose that (2) does not hold. Note that C is weakly sequentially compact. So there are $[w_n] \in \widetilde{W_t}$, $x \in C$, and a subsequence (w_{n_k}) of (w_n) such that

$$\lim_{k\to\infty} \|w_{n_k} - x\| \text{ exists} \quad \text{and} \quad \lim_{k\to\infty} \|w_{n_k} - x\| < t.$$

Then

$$\limsup_{k\to\infty} \|x_{n_k} - x\| \le \lim_{k\to\infty} \|w_{n_k} - x\| + \lim_{k\to\infty} \|x_{n_k} - w_{n_k}\| < 1,$$

a contradiction.

(3) follows from (2) and (c).

Proof of (4). By (b), we need only to show that for any subsequence (w_{n_k}) of (w_n) we have

$$\limsup_{k\to\infty}\limsup_{j\to\infty}\|w_{n_k}-w_{n_j}\|\ge t.$$

Suppose that it is not true. Then by passing to further subsequences of (w_{k_n}), we may assume that

$$\limsup_{k\to\infty}\limsup_{j\to\infty}\|w_{n_k}-w_{n_j}\|< t.$$

Let w be a weak limit point of (w_{n_k}). Then

$$\limsup_{k\to\infty}\|w_{n_k}-w\|\le\limsup_{k\to\infty}\limsup_{j\to\infty}\|w_{n_k}-w_{n_j}\|<t,$$

and

$$\lim_{k\to\infty}\|x_{n_k}-w\|\le\limsup_{k\to\infty}\|x_{n_k}-w_{n_k}\|+\limsup_{n\to\infty}\|w_{n_k}-w\|<1,$$

a contradiction.

Proof of (5). Suppose that (w_{n_k}) converges to w weakly. Then

$$\|w\|\le\limsup_{n\to\infty}\|x_{n_k}-w_{n_k}\|\le 1-t.$$

And

$$\liminf_{n\to\infty}\|x_n-w_n+w\|\ge\lim_{n\to\infty}\|x_n\|-\lim_{n\to\infty}\|w_n-w\|=1-t.$$

The proof is complete. □

Lemma 2.3 *For any $\delta>0$ and $0<t<1$, there are $[w_n]\in\widetilde{W_t}$, a subsequence (w_{n_k}) of (w_k), and a sequence $(w^*_{n_k})$ in X^* that satisfies the following conditions:*

(1) $\limsup_{n\to\infty}\|w_n\|\ge 1-\delta$.
(2) *(w_{n_k}) converges to w weakly and $1-t\ge\|w\|\ge 1-t-\delta$.*
(3) *For any $j\ne k$, $t+\delta>\|w_{n_k}-w_{n_j}\|\ge t-\delta$.*
(4) *For any $k\in\mathbb{N}$, $\|w^*_{n_k}\|=1$ and $w^*_{n_k}(w_{n_k})=\|w_{n_k}\|$.*
(5) *For any k, $1-t\ge w^*_{n_k}(w)\ge(1-t)-2\delta$.*
(6) *For any $j\ne k$, $w^*_{n_j}(w_{n_k})\ge 1-t-4\delta$.*

Proof Fix $\delta>0$ and $0<t<1$. By Lemma 2.1, there is an element $[w_n]$ in $\widetilde{W_t}$ such that $\limsup_n\|w_n\|>1-\delta$. By passing to a subsequence of (w_n), we may assume that (w_n) converges to w weakly and $1\ge\|w_n\|\ge 1-\delta$ for all $n\in\mathbb{N}$. Since $\lim_{n\to\infty}\|w_n-w\|=t$ and $\|[x_n]-[w_n]\|_{\widetilde{X}}=1-t$,

$$\begin{aligned}1-t&=\limsup_{n\to\infty}\|x_n-w_n\|\\&\ge\|w\|\ge\|[w_n]\|_{\widetilde{X}}-\|[w_n]-w\|_{\widetilde{X}}\ge 1-t-\delta.\end{aligned}$$

Note that for any subsequence (w_{n_k}) of (w_n),

$$\limsup_{k\to\infty}\limsup_{j\to\infty}\|w_{n_j}-w_{n_k}\|=t.$$

By passing to further subsequences of (w_n) and applying the diagonal method, there is a subsequence (w_{n_k}) of (w_n) such that

$$t+\delta>\|w_{n_j}-w_{n_k}\|\ge t-\delta\quad\text{for all } j\ne k.$$

Let $w^*_{n_k}$ be any support function of w_{n_k} (i.e., $\|w^*_{n_k}\|=1$ and $w^*_{n_k}(w_{n_k})=\|w_{n_k}\|$). For any $k\in\mathbb{N}$,

$$\begin{aligned}1-t\ge w^*_{n_k}(w)&=w^*_{n_k}(w-w_{n_k}+w_{n_k})\\&\ge w^*_{n_k}(w_{n_k})-\|w-w_{n_k}\|\\&\ge 1-\delta-t-\delta=1-t-2\delta.\end{aligned}$$

Thus, for any k,

$$1 - t - 2\delta \leq w_{n_k}^*(w) = \lim_{j\to\infty} w_{n_k}^*(w_{n_j}) \leq 1 - t.$$

By passing to further subsequence of (w_{n_k}), we may assume that if $k < j$, then

$$1 - t - 3\delta \leq w_{n_k}^*(w_{n_j}) \leq 1 - t + \delta.$$

Passing to subsequence of (w_{n_k}) and applying the diagonal method, we assume that for any j, the limit $\lim_{k\to\infty} w_{n_k}^*(w_{n_j})$ exists. Let w^* be any w^*-limit point of $(w_{n_k}^*)$. Then

$$\begin{aligned}
&1 - t - 2\delta \leq \lim_{k\to\infty} w_{n_k}^*(w) = w^*(w) \leq 1 - t + \delta;\\
&w^*(w) = \lim_{k\to\infty} w^*(w_{n_k});\\
&w^*(w_{n_k}) = \lim_{j\to\infty} w_{n_j}^*(w_{n_k}) \quad \text{for all } k.
\end{aligned}$$

First, by passing to subsequence of (w_{n_k}), we may assume that for all k,

$$1 - t - 3\delta \leq w^*(w_{n_k}) \leq 1 - t + 2\delta.$$

Then passing to further subsequences of (w_{n_k}) and applying the diagonal method again, we can assume that for any $k < j$,

$$1 - t - 4\delta \leq w_{n_j}^*(w_{n_k}) \leq 1 - t + 3\delta.$$

The proof is complete. □

We have the following theorem [5, Theorem 5].

Theorem 2.4 *Let X be a Banach space without the weak fixed point property. For any $\epsilon > 0$, there is an infinite subset A of the unit ball of X^* such that $A = -A$ and $\|x^* - y^*\| \geq 1 - \epsilon$ for any two distinct points x^*, y^* in A. Hence, if X is E-convex, then X has the fixed point property.*

Proof Let A be the set of the elements $w_{n_k}^*$ in Lemma 2.3. Then $\|w_{n_k}^*\| = 1$, and for any $j \neq k$,

$$2 \geq \|w_{n_k}^* + w_{n_j}^*\| \geq \frac{(w_{n_k}^* + w_{n_j}^*)(w)}{\|w\|} \geq \frac{2(1-t) - 4\delta}{1 - t + \delta},$$

and

$$\begin{aligned}
2 \geq \|w_{n_k}^* - w_{n_j}^*\| &\geq \frac{(w_{n_k}^* - w_{n_j}^*)(w_{n_k} - w_{n_j})}{\|w_{n_k} - w_{n_j}\|}\\
&= \frac{w_{n_k}^*(w_{n_k}) - w_{n_j}^*(w_{n_j}) - w_{n_j}^*(w_{n_k}) + w_{n_j}^*(w_{n_j})}{\|w_{n_k} - w_{n_j}\|}\\
&> \frac{2(1-\delta) - 2\big(1 - t + 4\delta\big)}{\|w_{n_k} - w_{n_j}\|} = \frac{2t - 10\delta}{t + \delta}.
\end{aligned}$$

Since δ is an arbitrary positive real, we have proved the theorem. □

3 Renorming and the weak fixed point property

In this section, we will show that every reflexive Banach space can be renormed to satisfy the fixed point property [3,4]. Let us recall the definition $R(X)$ again.

Let X be a Banach space. Let $\mathcal{A}$ be the set of all weakly null sequence (x_n) in X such that $\lim_{n\to\infty}\|x_n\| = 1$. $R(X)$ is defined by

$$R(X) = \sup\{\liminf \|x_n + x\| : (x_n) \in \mathcal{A},\ \|x\| = 1\}.$$

Let X, Y be Banach spaces. The product space $(X\oplus Y)_2$ is the space $X\oplus Y$ with the norm $\|(x, y)\|_{(X\oplus Y)_2} = \left(\|x\|^2 + \|y\|^2\right)^{1/2}$. The map $x \mapsto (x, 0)$ (respectively, $y \mapsto (0, y)$ from X (respectively, Y) to $(X \oplus Y)_2$ is an embedding. We should write the sets $\{(x, 0) : x \in X\}$ and $\{(0, y) : y \in Y\}$ as X and Y. First we need the following lemma.

Lemma 3.1 *Let $(X, \|\cdot\|_X)$ and $(Y, \|\cdot\|_Y)$ be two Banach spaces. Assume that $R(Y) < 2$. For any subspace Z of $(X \oplus Y)_2$, if $Z \cap X = \{0\}$, then Z has the weak fixed point property.*

Proof Suppose that Z does not have the weak fixed point property. Then there are a weakly compact convex subset C of X, a nonexpansive map $T : C \to C$, and a weak null sequence $\big(z_n = (x_n, y_n)\big)$ in C such that C is a T-invariant minimal set (so T has no fixed point), $\operatorname{diam}(C) = 1$, and (z_n) is an approximate fixed point sequence for T. By passing to further subsequences of (z_n), we may assume that for any $(x, y) \in C$, both $\lim_{n\to\infty}\|x_n - x\|_X$ and $\lim_{n\to\infty}\|y_n - y\|_Y$ exist. By the Goebel-Karlovitz Lemma, for any $z = (x, y) \in C$,

$$1 = \limsup_{n\to\infty}\|z_n - z\|_{(X\oplus Y)_2} = \Big(\lim_{n\to\infty}\|x_n - x\|_X^2 + \lim_{n\to\infty}\|y_n - y\|_Y\Big)^{1/2}.$$

Since ℓ_2 is uniformly convex, there is a constant $\alpha \le 1$ such that for any $z = (x, y) \in C$,

$$\lim_{n\to\infty}\|x_n - x\|_X = \alpha \text{and} \lim_{n\to\infty}\|y_n - y\|_Y = \sqrt{1-\alpha^2}.$$

We claim that (y_n) is not a null sequence. Suppose that the claim is not true. Then for any $z = (x, y) \in C$,

$$\sqrt{1-\alpha^2} = \lim_{n\to\infty}\|y_n - y\| = \|y\|,$$
$$\sqrt{1-\alpha^2} = \lim_{m\to\infty}\|y_n - y_m\| = \lim_{m\to\infty}\|y_m\| = 0.$$

This implies that $C \subseteq X$. This contradicts the assumption, $Z\cap X = \{0\}$. We have proved that there are $\beta, \epsilon > 0$ such that for any $z = (x, y) \in C$, if $\|Tz - z\|_{(X\oplus Y)_2} \le \epsilon$, then $\|y\| \ge \beta$. By Lemma 2.3, for each $n \in \mathbb{N}$, there are a sequence $\big(w_k^n = (x_k^n, y_k^n)\big)$ in C and $w^n = (x^n, y^n) \in C$ such that

(1) $\|Tw_k^n - w_k^n\|_{(X\oplus Y)_2} < \frac{1}{n}$ and $1 \ge \|w_k^n\|_{(X\oplus Y)_2} \ge 1 - \frac{1}{n}$.
(2) $\frac{1}{2} + \frac{1}{n} \ge \|w_k^n - w^n\|_{(X\oplus Y)_2} \ge \frac{1}{2} - \frac{1}{n}$ for all n and k.
(3) For each n, (w_k^n) converges to w^n weakly and $\frac{1}{2} \ge \|w^n\|_{(X\oplus Y)_2} \ge \frac{1}{2} - \frac{1}{n}$.

For any n, by passing to a further subsequence, we may assume that $\|y_k^n - y^n\|_Y \ge \frac{\beta}{4}$ for all k or $\|y_k^n - y^n\|_Y \le \frac{\beta}{4}$ for all k. Suppose that $\frac{1}{n} < \epsilon$. Then $\|y_k^n\|_Y \ge \beta$.

Case 1. There is an infinite subset A of $\mathbb{N}$ such that $\|y_k^n - y^n\|_Y < \frac{\beta}{4}$ for all n, k. Then for any $n \in A$, $\|y^n\|_Y \ge \|y_k^n\|_Y - \|y_k^n - y^n\|_Y \ge \frac{3\beta}{4}$. By the parallelogram law,

$$\begin{aligned}
&\|w^n + (w_k^n - w^n)\|_{(X\oplus Y)_2}^2\\
&= 2\|w^n\|_{(X\oplus Y)_2}^2 + 2\|w^n - w_k^n\|_{(X\oplus Y)_2} - \|2w^n - w_k^n\|_{(X\oplus Y)_2}^2\\
&\le 4\Big(\frac{1}{2} + \frac{1}{n}\Big)^2 - (\|y_k^n - y^n\|_Y - \|y^n\|_Y)^2\\
&\le 1 + \frac{4}{n} + \frac{4}{n^2} - \frac{\beta^2}{4}.
\end{aligned}$$

Fix k and let $n \in A$ approach to infinity. We have

$$1 = \lim_{n\to\infty, n\in A} \|w_n^k\|^2_{(X\oplus Y)_2} \le 1 - \frac{\beta^2}{4} < 1,$$

a contradiction. We will get a similar contradiction if there is an infinite subset A of $\mathbb{N}$ such that $\|y^n\|_Y \le \frac{\beta}{4}$ for all $n \in A$.

Case 2. There is an infinite subset A of $\mathbb{N}$ such that

$$\|y^n\|_Y \ge \frac{\beta}{4} \quad \text{and} \quad \|y_k^n - y^n\|_Y \ge \frac{\beta}{4}.$$

For each n, by passing to subsequences of $(w_k^n)_{k=1}^{\infty}$, we may assume that all limits $\lim_{k\to\infty} \|x_k^n - x^n\|_X$, $\lim_{k\to\infty} \|y_k^n\|_Y$, and $\lim_{k\to\infty} \|y_k^n - y^n\|_Y$ exist. Then if $n \in A$, then

$$\begin{aligned}
\lim_{k\to\infty} \|y_k^n\|_Y &= \lim_{k\to\infty} \|y^n + (y_k^n - y^n)\|_Y \\
&\le \left(\|y^n\|_Y - \frac{\beta}{4}\right) + \lim_{k\to\infty}\left(\|y_k^n - y^n\|_Y - \frac{\beta}{4}\right) + \frac{\beta}{4}\lim_{k\to\infty}\left(\frac{y^n}{\|y^n\|_Y} + \frac{y_k^n}{\|y_k^n\|_Y}\right) \\
&\le \left(\|y^n\|_Y - \frac{\beta}{4}\right) + \lim_{k\to\infty}\left(\|y_k^n - y^n\|_Y - \frac{\beta}{4}\right) + \frac{\beta R(X)}{4} \\
&= \|y^n\|_Y + \lim_{k\to\infty} \|y_n^k - y^n\|_Y - \frac{(2 - R(Y))\beta}{4},
\end{aligned}$$

and

$$\begin{aligned}
\lim_{k\to\infty} \|w_k^n\|^2_{(X\oplus Y)_2} = \lim_{k\to\infty} \|w^n + (w_k^n - w^n)\|^2_{(X\oplus Y)_2} &\le \left(\|x^n\|_X + \lim_{k\to\infty} \|x_k^n - x^n\|_X\right)^2 \\
&+ \left(\|y^n\|_Y + \lim_{k\to\infty} \|y_n^k - y^n\|_Y - \frac{(2 - R(Y))\beta}{4}\right)^2.
\end{aligned}$$

This implies that

$$1 = \lim_{n\to\infty} \lim_{k\to\infty} \|w_k^n\|^2_{(X\oplus Y)_2} \le 1 - \frac{(2 - R(Y))^2\beta^2}{16}.$$

We get a contradiction. So Z has the weak fixed point property. □

We have the following theorem.

Theorem 3.2 *Let X, Y be Banach spaces such that $R(Y) < 2$. Suppose that there is a one-to-one linear continuous map T from X to Y. Then for any $1 > \delta > 0$, there is an equivalent norm $|\cdot|_X$ of X such that $d\big((X, \|\cdot\|_X), (X, |\cdot|_X)\big) \le 1 + \delta$ and $(X, |\cdot|_X)$ satisfies the weak fixed point property.*

Proof Let I denote the identity map from X to X and T a one-to one linear continuous map from X to Y. Replacing T by $\frac{\delta T}{\|T\|}$. We may assume that $\|T\| \le \delta$. Then $I + T$ is a bounded linear mapping from X to $(X \oplus Y)_2$. Then for any $x \in X$,

$$\|x\| \le \|(I + T)(x)\|_{(X\oplus Y)_2} \le \sqrt{1 + \delta^2}.$$

It is easy to see that $(I + T)(X) \cap X = \{0\}$. The proof is complete. □

It is easy to see for any set Γ, $R\big(c_0(\Gamma)\big) = 1$. It is known that for any reflexive Banach space X, X can be embedded in $c_0(\Gamma)$ for some Γ [23]. We have the following corollary.

Corollary 3.3 [3, Corollary] *Every reflexive Banach space has an equivalent norm to satisfy the fixed point property.*

Theorem 3.4 *Let X, Y be two Banach spaces such that X has the weak fixed point property and $R(Y) < 2$. Then $(X \oplus Y)_2$ has the weak fixed property.*

Proof Let C be a weakly compact, convex subset of $X \oplus Y$ and $T : C \to C$ a nonexpansive map. Without loss of generality, we assume that C is minimal and there is an approximate fixed sequence (z_n) (for T) that converges to 0. We have shown that either $C \subset X$ or T has a fixed point. Since X has the weak fixed point property, T has at least one fixed point. □

4 Stability of the fixed point property

Let X, Y be two isomorphic Banach spaces. The Banach–Mazur distance $d(X, Y)$ from X to Y is defined by

$$d(X, Y) = \inf \left\{ \|S\| \cdot \|S^{-1}\| : S \text{ is an isomorphism from } X \text{ onto } Y \right\}.$$

If X is not isomorphic to Y, then we say $d(X, Y) = \infty$. Let C be a closed, bounded, convex subset of a Banach space $(X, \|\cdot\|)$. A map $T : C \to C$ is said to be *uniformly Lipschitz* if there is M such that for any $n \in \mathbb{N}$ and $x, y \in C$,

$$\|T^n x - T^n y\| \le M\|x - y\|.$$

We denote the best constant by $\|T\|_{Lip}$. Let $|\cdot|$ be an equivalent norm on X such that $d\big((X, \|\cdot\|), (X, |\cdot|)\big) \le B$. If $T : (C, |\cdot|) \to (C, |\cdot|)$ is nonexpansive, then $T : (C, \|\cdot\|) \to (C, \|\cdot\|)$ is uniformly Lipschitz and the uniformly Lipschitz constant is less than or equal to B. It is known that if X is uniformly convex or X has uniformly normal structure, then there is $C > 0$ such that for any weakly compact subset C and any uniformly Lipschitz map T with $\|T\|_{Lip} < C$, T has a fixed point. This implies that for any uniformly convex Banach space X, there is $C > 1$ such that for any Banach space Y, if $d(X, Y) < C$, then Y has the weak fixed point property. In this section, we try to improve the constant for ℓ_p, $1 < p < \infty$.

Let $(X, |\cdot|)$ be a Banach space such that $(X, \ell_p) < B_p$. Without loss of generality, we may assume that

$$\|x\|_p \le |x| \le B_p \|x\|_p \quad \text{for any } x \in X.$$

We shall show that there is a constant C_p such that if $(X, |\cdot|)$ does not have the (weak) fixed point property, then $B_p > C_p^{1/p}$.

Suppose that $(X, |\cdot|)$ does not have the fixed point property. Let C be a weakly compact convex subset of X and $T : C \to C$ a nonexpansive mapping such that $\operatorname{diam}(C) = 1$ and C is minimal. By translation, we may assume that there is an approximate fixed point sequence (x_n) for T that converges to 0 weakly. First, we need the following lemma [10].

Lemma 4.1 *For any $0 < t < 1$, there are $[w_n]$ in $\widetilde{W_t}$ and a subsequence (w_{n_k}) of (w_n) that satisfy the following conditions:*

(1) (w_{n_k}) *converges to* w;

(2) $(1 - t + \delta)^p \ge \left(\frac{1-t-\delta}{B_p}\right)^p + \|w\|_p^p$ *and* $\left(\frac{1-\delta}{B_p}\right)^p \le \frac{(t+\delta)^p}{2} + \|w\|_p^p$.

Proof Fix $t > \delta > 0$. By Lemma 2.2, there is a sequence (w_{n_k}) in C such that

(a) $|w_{n_k}| \ge 1 - \delta$;

(b) (w_{n_k}) converges to w weakly and $1 - t \ge |w| \ge 1 - t - \delta$;

(c) for all k,

$$|w_{n_k} - w| \le t + \delta;$$
$$1 - t + \delta \ge |x_{n_k} - w_{n_k}|;$$
$$|x_{n_k} - w_{n_k} + w| \ge 1 - t - \delta.$$

By passing to further subsequences of (w_{n_k}), we may assume that all limits

$$\lim_{k\to\infty} \|w_k - w\|_p, \quad \lim_{k\to\infty} \|w_{n_k} - w - x_{n_k}\|_p,$$
$$\lim_{k\to\infty} \|x_{n_k} - w_{n_k}\|_p, \quad \lim_{k\to\infty} \|w_{n_k}\|_p$$

exist. It is know that if (z_n) is a weakly null sequence in ℓ_p and if $z \in \ell_p$, then

$$\limsup_{n\to\infty} \|z_n - z\|_p^p = \|z\|_p^p + \limsup_{n\to\infty} \|z_n\|_p^p.$$

Since (x_n) and (w_{n_k}) are weakly null sequences in ℓ_p, we have

$$\begin{aligned}(1-t+\delta)^p &\geq \limsup_{k\to\infty} |x_{n_k}-w_{n_k}|^p \geq \lim_{k\to\infty} \|x_{n_k}-w_{n_k}\|_p^p \\ &= \|w\|_p^p + \lim_{k\to\infty} \|x_{n_k}-w_{n_k}+w\|_p^p \\ &\geq \|w\|_p^p + \lim_{k\to\infty} \frac{|x_{n_k}-w_{n_k}+w|^p}{B_p^p} \\ &\geq \|w\|_p^p + \frac{(1-t-\delta)^p}{B_p^p}; \\ \lim_{k\to\infty} 2\|w_{n_k}-w\|_p^p &= \lim_{k\to\infty} \|w_{n_k}-w\|_p^p + \lim_{j\to\infty} \|w_{n_j}-w\|_p^p \\ &= \lim_{k\to\infty}\lim_{j\to\infty} \|w_{n_k}-w_{n_j}\|_p^p \\ &\leq \limsup_{k\to\infty}\limsup_{j\to\infty} |w_{n_k}-w_{n_j}|^p \leq (t+\delta)^p.\end{aligned}$$

We have proved that $\lim_{k\to\infty} \|w_{n_k}-w\|_p^p \leq \frac{(t+\delta)^p}{2}$. So

$$\left(\frac{1-\delta}{B_p}\right)^p \leq \lim_{k\to\infty} \|w_{n_k}\|_p^p = \|w\|_p^p + \lim_{k\to\infty} \|w_{n_k}-w\|_p^p \leq \|w\|_p^p + \frac{(t+\delta)^p}{2}.$$

The proof is complete. □

Theorem 4.2 *For* $1 < p < \infty$, *let* $C_p > 1$ *be the smallest positive solution of the equation*

$$C(C-1) = [C^{1/(p-1)} + (2C-2)^{1/(p-1)}]^{p-1}.$$

If the Banach–Mazur distance from $(X, |\cdot|)$ *to* ℓ_p *is less than* $(C_p)^{1/p}$, *then* X *has the fixed point property.*

Proof Suppose that $(X, |\cdot|)$ does not have the weak fixed point property. By Lemma 4.1, for any $0 < \delta < t < 1$, there is w such that

$$(1-t+\delta)^p \geq \left(\frac{1-t-\delta}{B_p}\right)^p + \|w\|_p^p;$$
$$\left(\frac{1-\delta}{B_p}\right)^p \leq \frac{(t+\delta)^p}{2} + \|w\|_p^p.$$

So

$$(1-t+\delta)^p - \left(\frac{1-t-\delta}{B_p}\right)^p \geq \|w\|_p^p \geq \frac{(1-\delta)^p}{B_p^p} - \frac{(t+\delta)^p}{2}.$$

Let δ approach to 0. We have

$$(1-t)^p - \left(\frac{1-t}{B_p}\right)^p - \frac{1}{B_p^p} + \frac{t^p}{2} \geq 0$$

and

$$(B_p^p - 1)(1-t)^p - 1 + \frac{B_p^p t^p}{2} \geq 0.$$

Let $C = B_p^p$ and $t = \frac{(2C-2)^{1/(p-1)}}{C^{1/(p-1)}+(2C-2)^{1/(p-1)}}$. Then

$$C(C-1)\left(\frac{C^{1/(p-1)} + (2C-2)^{1/(p-1)}}{\left(C^{1/(p-1)} + (2C-2)^{1/(p-1)}\right)^p}\right) - 1 \geq 0.$$

Hence

$$C(C-1) \geq \left(C^{1/(p-1)} + (2C-2)^{1/(p-1)}\right)^{p-1}$$

and $B_p^p \geq C_p$. The proof is complete. □

Let T be the nonexpansive map in Lemma 4.1 and Theorem 4.2 and let

$$D_p = \inf\{\liminf_{n\to\infty} \|y_n - y\|_p : (y_n) \text{is an approximate fixed sequence}$$
$$\text{for } T \text{ and } (y_n) \text{ converges to } y \text{ weakly}\}.$$

Clearly, $D_p \geq \frac{1}{B_p}$. It is known that ℓ_p, $1 \leq p < \infty$, has the Opial property. For any $\epsilon > 0$ and any $x \in C$, there is $\delta > 0$ such that if $\widetilde{w} = [w_n] \in \widetilde{C}$ with $\|\widetilde{w} - \widetilde{T}\,\widetilde{w}\|_{\widetilde{X}} < \delta$, then

$$\liminf_{n\to\infty} \|w_n - x\|_p \geq D_p - \epsilon.$$

Hence if $D_p > \frac{1}{B_p}$, then we can improve the constant in Theorem 4.2. We claim that

$$D_2^2 \geq \frac{B_2^2 - 1}{B_2^2(B_2^2 - 2)}.$$

Suppose that the claim has been proved. Then for any $\epsilon > 0$ and x, there is $\delta > 0$ such that if $\widetilde{w} = [w_n] \in \widetilde{C}$ and if $\|\widetilde{w} - \widetilde{T}\widetilde{w}\|_{\widetilde{X}} < \delta$, then

$$\liminf_{n\to\infty} \|w_n - x\|_2 \geq D_2 - \epsilon.$$

By the proof of Lemma 4.1, we have the following inequalities: for any $\epsilon, \delta > 0$, there is $w \in C$ such that

$$(1 - t + \delta)^2 \geq \|w\|_2^2 + \frac{(1 - t - \delta)^2}{B_2^2};$$
$$(D_2 - \epsilon)^2 \leq \lim_{k\to\infty} \|w_{n_k}\|_2^2 \leq \|w\|_2^2 + \frac{(t + \delta)^2}{2}.$$

We have

$$(1 - t + \delta)^2 - \Big(\frac{1 - t - \delta}{B_2}\Big)^2 \geq (D_2 - \epsilon)^2 - \frac{(t - \delta)^2}{2}.$$

Let δ and ϵ approach to 0 and we have

$$(1 - t)^2\Big(1 - \frac{1}{B_2^2}\Big) \geq \frac{1}{D_2^2} - \frac{t^2}{2} \geq \frac{B_2^2 - 1}{B_2^2(B_2^2 - 2)} - \frac{t^2}{2}.$$

Let $t = \frac{2(B^2-1)}{3B^2-2}$. We have

$$\begin{aligned}
&\frac{(B_2^2)^2}{(3B_2^2 - 2)^2} \cdot \frac{B_2^2 - 1}{B_2^2} - \frac{B_2^2 - 1}{B_2^2(B_2^2 - 2)} + \frac{4(B_2^2 - 1)^2}{2(3B_2^2 - 2)^2}\\
&= \frac{(B_2^2 - 1)(B_2^2 + 2(B_2^2 - 1))}{(3B^2 - 2)^2} - \frac{B_2^2 - 1}{B_2^2(B_2^2 - 2)}\\
&= \frac{B_2^2 - 1}{3B^2 - 2} - \frac{B_2^2 - 1}{B_2^2(B_2^2 - 2)} = (B_2^2 - 1)\Big(\frac{1}{3B_2^2 - 2} - \frac{1}{B_2^2(B_2^2 - 2)}\Big) \geq 0.
\end{aligned}$$

Since $B_2 > 1$, we have

$$B_2^4 - 5B_2^2 + 2 \geq 0.$$

This implies that $B_2 \geq \sqrt{\frac{5+\sqrt{17}}{2}}$. We have the following theorem [18].

Theorem 4.3 *Let X be a Banach space such that $d(X, \ell_2) < \sqrt{\frac{5+\sqrt{17}}{2}}$. Then X has the fixed point property.*

Proof of the claim. First, we need the following facts:

(1) The ultrapower of a Hilbert space is a Hilbert space. We denote the ultrapowers of $(X, |\cdot|)$ and $(\ell_2, \|\cdot\|_2)$ by $(\widehat{X}, \|\cdot\|_{\widehat{X}})$ and $(H, \|\cdot\|_2)$, respectively. Since $\|x\|_2 \le |x| \le B_2\|x\|_2$ for any $x \in X$,

$$\|\widehat{x}|_2 \le \|\widehat{x}\|_{\widehat{X}} \le B_2\|\widehat{x}\|_2 \quad \text{for any } \widehat{x} \in \widehat{X}.$$

(2) Since C is weakly compact, the weak limit $w = \text{w-}\lim_{n\to\mathcal{U}} w_n$ exists for any $[w_n] \in \widehat{C}$.

(3) For any $[z_n] \in \widehat{X} = H$ and any $z \in X = \ell_2$, if $\text{w-}\lim_{n\to\mathcal{U}} z_n = 0$, then

$$\|[z_n] - z\|_2^2 = \|[z_n]\|_2^2 + \|z\|_2^2.$$

(4) The generalization of the parallelogram law: Given $x, y \in \ell_2$ and given $\lambda \in (0, 1)$, we have the identity

$$\begin{aligned}
&\|\lambda x + (1-\lambda)y\|_2^2\\
&\quad = \lambda^2\|x\|_2^2 - 2\lambda(1-\lambda)\langle x, y\rangle + (1-\lambda)^2\|y\|_2^2\\
&\quad = \lambda\|x\|_2^2 + (1-\lambda)\|y\|_2^2 - \lambda(1-\lambda)\big(2\langle x, y\rangle - \|x\|_2^2 - \|y\|_2^2\big)\\
&\quad = \lambda\|x\|_2^2 + (1-\lambda)\|y\|_2^2 - \lambda(1-\lambda)\|x-y\|_2^2.
\end{aligned}$$

(5) Let $\widehat{T}$ be the nonexpansive map from $\widehat{C}$ to $\widehat{C}$ defined by

$$\widehat{T}[z_n] = [T z_n].$$

Let $\epsilon > 0$. By the definition of D_2 and by translation if necessary, there is a fixed point $\widehat{x} = [x_n] \in \widehat{C}$ such that $\|\widehat{x}\|_{\widehat{X}} \le D_2 + \epsilon$ and $\text{w-}\lim_{n\to\mathcal{U}} x_n = 0$. For each $0 < t < 1$, let $\widehat{W_t}$ be the minimal $\widehat{T}$-invariant set that contains $t\widehat{x}$. By the proofs in section 2 and the remark after the definition of D_p, there is $\widehat{w_t} = [w_n] \in \widehat{W_t}$ such that

$$\begin{aligned}
&\|\widehat{w_t}\|_{\widehat{X}} \ge 1-\epsilon,\\
&\|\widehat{w_t}\|_2 \ge D_2-\epsilon,\\
&\|\widehat{w_t} - \widehat{x}\|_{\widehat{X}} = 1-t,\\
&\|\widehat{w_t} - w\|_{\widehat{X}} = t \quad \text{where } w = \text{w} - \lim_{n\to\mathcal{U}} w_n,\\
&\|w\|_{\widehat{X}} \ge \|\widehat{w_t}\|_{\widehat{X}} - \|\widehat{w_t} - w\|_{\widehat{X}} \ge 1-\epsilon-t.
\end{aligned}$$

For any $0 < \lambda < 1$, we need to estimate the norm $\|\lambda\widehat{w_t} + (1-\lambda)t\widehat{x} - \widehat{x}\|_2$. Note that

$$\text{w} - \lim_{n\to\mathcal{U}} \lambda w_n + (1-\lambda)t x_n - x_n = \lambda w$$

and

$$\begin{aligned}
&\|\lambda(\widehat{w_t} - w) + (1-\lambda)t\widehat{x} - \widehat{x}\|_{\widehat{X}}\\
&\ge \|\widehat{x}\|_{\widehat{X}} - (1-\lambda)t\|\widehat{x}\|_{\widehat{X}} - \lambda\|\widehat{w_t} - w\|_{\widehat{X}}\\
&= 1 - (1-\lambda)t - \lambda t = 1-t.
\end{aligned}$$

We have

$$\begin{aligned}
&\|\lambda\widehat{w_t} + (1-\lambda)t\widehat{x} - \widehat{x}\|_2^2\\
&= \|\lambda(\widehat{w_t} - w) + (1-\lambda)t\widehat{x} - \widehat{x}\|_2^2 + \lambda^2\|w\|_2^2\\
&\ge \frac{1}{B_2^2}\big(\|\lambda(\widehat{w_t} - w) + (1-\lambda)t\widehat{x} - \widehat{x}\|_{\widehat{X}}^2 + \lambda^2\|w\|_{\widehat{X}}^2\big)\\
&= \frac{1}{B_2^2}\big(\lambda^2(1-t-\epsilon)^2 + (1-t)^2\big).
\end{aligned}$$

By the generalized parallelogram law,

$$\begin{aligned}
&\|\lambda\widehat{w_t} + (1-\lambda)t\widehat{x} - \widehat{x}\|_2 \\
&= \|\lambda(\widehat{w_t} - \widehat{x}) - (1-\lambda)(1-t)\widehat{x}\|_2 \\
&= \lambda\|\widehat{w_t} - \widehat{x}\|_2^2 + (1-\lambda)(1-t)^2\|\widehat{x}\|_2^2 - \lambda(1-\lambda)\|\widehat{w_t} - t\widehat{x}\|_2^2;
\end{aligned}$$

and

$$\begin{aligned}
&\|\widehat{w_t} - t\widehat{x}\|_2^2 \\
&= \|t(\widehat{w_t} - \widehat{x}) + (1-t)\widehat{w_t}\|_2^2 \\
&= t\|\widehat{w_t} - \widehat{x}\|_2^2 + (1-t)\|\widehat{w_t}\|_2^2 - t(1-t)\|\widehat{x}\|_2^2.
\end{aligned}$$

Combining those two inequalities, we have

$$\begin{aligned}
&\|\lambda\widehat{w_t} + (1-\lambda)t\widehat{x} - \widehat{x}\|_2^2 \\
&= \lambda(1-(1-\lambda)t)\|\widehat{w_t} - \widehat{x}\|_2^2 \\
&\quad + (1-\lambda)(1-t)(1-t+\lambda t)\|\widehat{x}\|_2^2 - \lambda(1-\lambda)(1-t)\|\widehat{w}\|_2^2 \\
&\le \lambda(1-(1-\lambda)t)\|\widehat{w_t} - \widehat{x}\|_{\widehat{X}}^2 \\
&\quad \times (1-\lambda)(1-t)(1-t+\lambda t)(D_2+\epsilon)^2 - \lambda(1-\lambda)(1-t)(D_2-\epsilon)^2 \\
&= \lambda(1-(1-\lambda)t)(1-t)^2 \\
&\quad + (1-\lambda)(1-t)(1-t+\lambda t)(D_2+\epsilon)^2 - \lambda(1-\lambda)(1-t)(D_2-\epsilon)^2.
\end{aligned}$$

Put two inequalities together and let $\epsilon \downarrow 0$. We obtain

$$\frac{(1-t)^2(1+\lambda^2)}{B_2^2} \le \lambda(1-(1-\lambda)t)(1-t)^2 + (1-\lambda)^2(1-t)^2 D_2^2.$$

Divide both side by $(1-t)^2$ and then let $t \uparrow 1$. We have

$$\frac{1+\lambda^2}{B_2^2} \le \lambda^2 + (1-\lambda)^2 D_2^2.$$

Note that $B_2 \ge \sqrt{2}$. Let $\lambda = \frac{1}{B_2^2-1} \le 1$. We get

$$D_2^2 \ge \frac{B_2^2-1}{B^2(B_2^2-2)}.$$

The proof is complete. □

5 Remarks and open questions

Since L_1 does not have the weak fixed point property, it is natural to ask the following question:

Question 1 *Let X be a reflexive (or superreflexive) Banach space. Does X have the fixed point property?*

More specifically we may ask the following two questions:

Question 2 *Does every reflexive Banach lattice have the fixed point property?*

Question 3 *Suppose that X is isomorphic to an L_p-space for some $1 < p < \infty$. Does X have the fixed point property? In particular, does X have the fixed point property if X is isomorphic to a Hilbert space?*

It is known that if E is a uniformly monotone Banach space (for example, X is L_p-space for some $1 \leq p < \infty$), then every superreflexive subspace of X has the fixed point property. But we do not know whether every superreflexive Banach lattice has the fixed point property or not.

In [14], it has been proved that if X has an unconditional basis with the unconditional constant less than $(\sqrt{33} - 3)/2$, then X has the weak fixed point property. We may ask the following question.

Question 4 *Let X be a Banach space with an unconditional basis. Does X have the weak fixed point property?*

Question 5 *Let C be a weakly compact convex subset of c_0. Does every uniformly Lipschitz map $T : C \to C$ have a fixed point?*

Let $(T, \|\cdot\|_T)$ be the Tsirelson space (for definition see [13, p. 95] and let $|\cdot|_T$ and $|\cdot|_{T^*}$ be the equivalent norms on T and T^* defined by

$$|x|_T = \max\{\|x^+\|_T, \|x^-\|_T\} \quad \text{for all } x \in T,$$
$$|x|_{T^*} = \|x^+\|_{T^*} + \|x^-\|_{T^*} \quad \text{for all } x \in T^*.$$

It is known that $(X^*, |\cdot|_{T^*})$ has the fixed point property. But we do not know whether $(T, |\cdot|_T)$ has the fixed property or not.

It is known that there is an equivalent norm on ℓ_1 that has the fixed point property [16]. But we do not know the answer of the following questions.

Question 6 *Is there an equivalent norm of c_0 that has the fixed point property? Is there an equivalent norm on James space that has the fixed point property?*

Question 7 *Is there an equivalent norm on L_1 (respectively, the trace class C_1) that has the fixed point property?*

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